Dr.-Ing. Michael Hernla
Sonnenplatz 13, 44137 Dortmund
Telefon 0231 136010
michael.hernla@t-online.de
www.dr-hernla.de

DR.-ING.
MICHAEL HERNLA

Uncertainty of coordinate measurements

Comparison of various methods of uncertainty evaluation

Abstract: *The number and positions of the measured points both have a great effect on the uncertainty of coordinate measurements, as may be illustrated by the confidence intervals of the geometric features. The procedures of uncertainty assessment recently discussed are either based on repeated measurements or on the simulation of errors of measurements respectively. Comparing these procedures with the mathematically exact solution, the advantages and disadvantages of the methods they are based on may be clearly identified. Only the exact solution may be used to assess the effects of the various influence quantities on the uncertainty.*

1. Fundamental principles of uncertainty evaluation

Today, the *Guide to the Expression of Uncertainty in Measurement (GUM)* [1] is the internationally accepted basis to assess the quality of measurements and to guarantee their comparability. The main requirements for the evaluation and documentation of the uncertainty are:

- Statement of the mathematical function $Y=f(X_i)$ for the relationship between the measurand Y and the influence quantities X_i representing the essential uncertainty components
- Determination of the best estimates x_i of the influence quantities X_i by statistical analysis or otherwise
- Determination of the standard uncertainties $u(x_i)$ of the influence quantities X_i by statistical analysis or otherwise
- Calculation of the covariances of potentially correlated influence quantities
- Calculation of the measurement result y as the value of the measurand Y derived from the function $Y=f(X_i)$ using the best estimates x_i of the influence quantities X_i
- Determination of the combined standard uncertainty $u_c(y)$ of the result from the standard uncertainties $u(x_i)$ and from the covariances
- Expression of the expanded uncertainty $U=k \cdot u_c(y)$; usually the expansion factor k=2 for a level of confidence of about P=95% is used
- Documentation of the result y with the combined standard uncertainty $u_c(y)$ or the expanded uncertainty U with a description of how y and $u_c(y)$ or U were determined

At first, the implementation of these requirements is a question of practice. The main challenge lies in how to establish the mathematical model of the measurement. Provided that it exists, the essential influence quantities may be easily recognized by the individual uncertainty components neglecting the others.

In coordinate measurement such detailed uncertainty determinations have not previously been carried out, because they are slightly more complicated. A special problem is the calculation of the uncertainty of the least square elements, which may usually not be expressed as a closed solution. However, the principles of the *GUM* are valid here, too (see [1], items 3.1 and C.3.5). The interesting thing about it is that the covariances between the individually resulting parameters of the least squares features are significant and generally not negligible.

2. Coordinate measurements

Several ways of calculating “best-fit” associated features are used in the software of most modern coordinate measuring machines (CMM). The most important of them is the method of least squares, calculating ideal geometric associated features from a relatively large number of measured points. The decisive advantage of this method is the excellent stability of the results compensating the errors of the individual points. It was established by C. F. GAUSS more than 200 years ago, and it is still today the fundamental method in all scientific and technical measurements.

The method of least squares und its uncertainty are generally described e.g. in [2] and within the German standard DIN 1319-4 [3]. Its application in coordinate measurement has been published e.g. in [4] and [5].

In least squares calculations, a linear homogenous equation system

$$\mathbf{M}\cdot\mathbf{x} = \mathbf{v} \qquad \text{rsp. } \mathbf{x} = \mathbf{M}^{-1}\cdot\mathbf{v} \qquad \text{with } \mathbf{M} = \mathbf{A}^T\mathbf{A} \tag{1}$$

is to be resolved. The matrix **M** of normal equations (rsp. the design matrix **A** [6]) contain the positions of the individually measured points on the surface of the geometric feature, the vector **v** the deviations of these points from the ideally geometric feature, and ***x*** being the vector of the solutions of the equation system as the individually resulting parameters. The matrix $\mathbf{M}^{-1}$ inverse to **M** is often indicated by the letter **Q**. Its multiplication with the variance σ^2 of the random and independent errors of the measured points from the ideally geometric feature results in the covariance matrix **S**, describing the mutual interdependence of the parameters of the geometric features as the measurands:

$$\mathbf{S} = \mathbf{M}^{-1}\cdot\sigma^2 = \mathbf{Q}\cdot\sigma^2 \tag{2}$$

The variances and covariances of the individually resulting parameters are calculated from the elements of the covariance matrix [2-5]. The variance σ^2 in first approximation may be estimated by the deviations of the measured points from the least squares feature. Should these deviations contain systematic components (e.g. because of local deviations of form on the surface), they may be separated from the random components, thus reducing the variance σ^2 and the uncertainties [3-5, 7]. The variances and covariances calculated in this way may be used in further uncertainty evaluations of coordinate measurements according to the *GUM*.

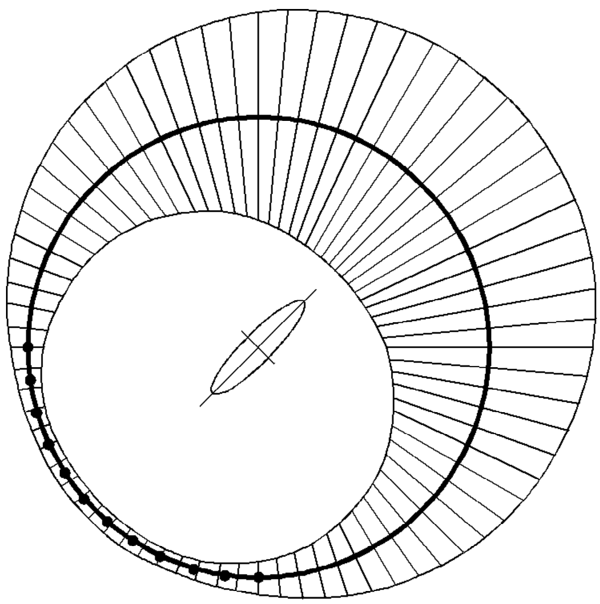
3. Least squares circle

At the least squares circle (with ϕ_i being the polar angels of the *n* measured points and the parameters *x*, *y* and *r*), the matrix **M** of normal equations results in:

$$\mathbf{M} = \begin{pmatrix} \sum\cos^2\phi_i & \sum\cos\phi_i\sin\phi_i & \sum\cos\phi_i \\ \sum\cos\phi_i\sin\phi_i & \sum\sin^2\phi_i & \sum\sin\phi_i \\ \sum\cos\phi_i & \sum\sin\phi_i & \sum 1 \end{pmatrix} \tag{3}$$

The dependence of the uncertainty of the individually resulting parameters on the positions of the measured points can be clearly illustrated. The confidence interval is small in the region of measured points capturing the real surface, and becomes wide with increasing distance to this section (Fig. 1). The confidence interval of the centre of the circle is an ellipse with a characteristic orientation, and the uncertainty of the diameter is remarkable, too.

Fig. 1:
Confidence intervals of the centre and the line of a circle with 12 points arranged at a sector with an angle of 90°; the uncertainty of the least squares circle is determined by the position of the measured points



Ideally, the measured points may be positioned in equal distances around the entire circumference. Then, the confidence interval of the circle's line is equally slim at every point, and that of the centre point a circle in itself (Fig. 2). In this case the covariances disappear, and the standard uncertainties u_M of the centre point and u_D of the diameter merely depend on the number n of points and the standard deviation s of the random and independent errors of the least squares circle:

$$u_X = u_Y = u_M = \sqrt{\frac{2}{n}} \cdot s\,, \qquad u_D = \frac{2}{\sqrt{n}} \cdot s \tag{4}$$

For all other patterns of measured points the variances and covariances have to be calculated individually.

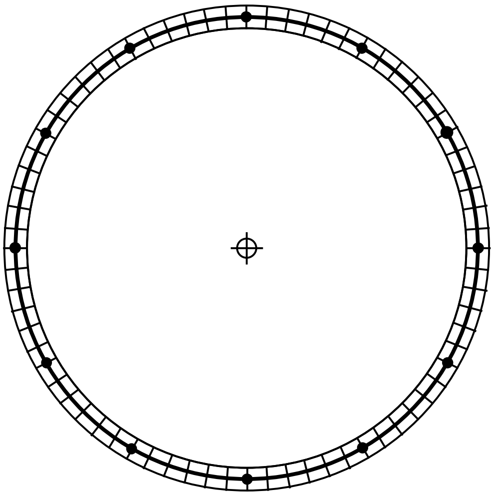


Fig. 2:
Confidence intervals of the centre and the line of a circle; the least uncertainty occurs when using 12 points arranged at equal distances around the whole circumference (same scale like in Fig. 1)

4. Other geometric features

Similar to the circle, the variances and covariances may be calculated for other geometric features, and the confidence intervals may be depicted, too. In probe system qualification for example, two points are often probed in the direction of the stylus shaft on the upper vertex, and four other points are positioned on the equator (Fig. 3). The standard uncertainties of both the stylus tip diameter and that of the centre point may then be calculated according to equation (4), too.
The confidence interval of the centre point is itself a sphere. The confidence interval of the upper hemisphere is almost constant. It becomes wider in the lower, because, similar to Fig. 1, there are no measured points.

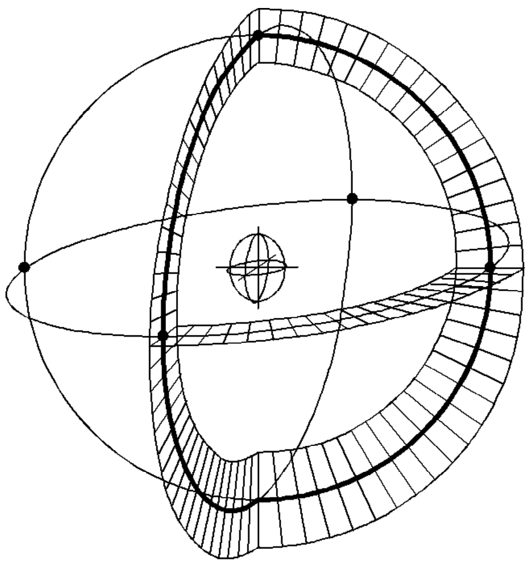


Fig. 3:
Confidence interval of a least squares sphere with six measured points, two of which are placed on the upper vertex

5. Uncertainty of a diameter

With the example of a measured diameter of a circle, the uncertainty of measurement shall be evaluated according to the *GUM*, taking into account the following influence quantities:

- Probing of the object's surface with eight equally distanced points
- Probe system qualification with six points according to Fig. 3
- Calibration of the spherical material standard according to its calibration certificate
- Temperatures of object, scale and material standard according to [8]
- Estimated geometrical errors of the CMM

In probe system qualification, the stylus tip diameter is calculated as the difference between the diameters D_E of the least squares sphere through the centre points of the stylus tip and D_C of the material standard. When measuring the object, the diameter D_W of the circle is calculated with the help of the centre points of the stylus tip. In case of an outside measurement the stylus tip diameter has to be subtracted, or added to the inside respectively. The mathematical model of a drill hole diameter reads as follows:

$$D = D_W + (D_E - D_C) \tag{5}$$

Furthermore, the influences of the temperatures and of the geometrical errors have to be taken into account. Using the terms from Table 1, the complete mathematical model results in:

$$D = \{ (D_W+D_E)*[1+\alpha_S(t_S-20°)] - D_C*[1-\alpha_C(t_C-20°)] \} *[1-\alpha_W(t_W-20°)] - \Delta D \tag{6}$$

The diameters D_W, D_E and D_C are calculated inside in the CMM and might not be known exactly to the user. The uncertainty may be calculated using the nominal values.

Table 1: Uncertainty budget of a drill hole diameter with the influence quantities:

- D_W Diameter of the least squares circle of the object
- D_E Diameter of the least squares sphere in probe system qualification
- D_C Calibrated diameter of the material standard
- α_W Thermal length expansion coefficient of the object (Steel)
- t_W Temperature of the object
- α_S Thermal length expansion coefficient of the scales (Float glass)
- t_S Temperature of the scales (Mean values of both axes)
- α_C Thermal length expansion coefficient of the material standard (Steel)
- t_C Temperature of the material standard
- ΔD Geometrical error of the CMM with MPE_E=(2+L/500) µm (L in mm) and L=D; limiting value a=D/500 µm

Influence Quantity	Unit	Best estimate value	Number of measured points rsp. distribution	Standard deviation rsp. limit	Standard uncertainty	Sensitivity coefficient	Uncertainty component
X_i	$[X_i]$	x_i	n_i	s_i rsp. a_i	$u(x_i)$	c_i	$u_i(y)$
D_W	mm	90	8	0.002	0.0014	1.0	0.0014
D_E	mm	40	6	0.001	0.0008	1.0	0.0008
D_C	mm	30	Normal	0.0004	0.0002	-1.0	-0.0002
α_W	µm/m/K	12	Normal	2.4	1.20	0.0000	0.0000
t_W	°C	20	Normal	1	0.50	-0.0012	-0.0006
α_S	µm/m/K	7,8	Normal	0.5	0.25	0.0000	0.0000
t_S	°C	20	Normal	1	0.50	0.0010	0.0005
α_C	µm/m/K	11	Normal	2.2	1.10	0.0000	0.0000
t_C	°C	20	Normal	1	0.50	0.0003	0.0002
ΔD	mm	0	Normal	0.0002	0.0001	-1.0	-0.0001
D	mm	100,0000			Combined standard uncertainty:	$u_c(y)$ =	0.0018
					Effective degrees of freedom:	ν_{eff} =	11.1
					Expansion factor:	k =	2.20
					Expanded uncertainty (P=95%):	U =	0.0040

For the temperature of the scale, the mean value of the two coordinate axes of that plane is used, in which the circle is to be measured. The geometrical error of the CMM is not known (ΔD=0), but it may be estimated as follows.

6. Estimation of the geometrical errors of the CMM

The most well known geometrical error of coordinate measuring machines is that of indication for size measurement [9]. It is usually verified by measuring gauge blocks or step gauges [10] with one probing point at every plane. The maximum permissible error MPE_E of indication for size measurements is usually stated in the form:

$$MPE_E = (A + \frac{L}{K})\ \mu\text{m} \qquad \text{(measured size } L \text{ in mm)} \qquad (7)$$

The constant A limits the probing uncertainty of the surface of the gauge and the uncertainty of the tip diameter. The length dependent component L/K limits the geometrical errors of the CMM, e.g. that of squareness, straightness and of the scales [11].
Measuring a diameter, usually more than two points are probed. According to (4), the standard uncertainty of the mean diameter D_W depends directly on the number of points, if the deviations are random and uncorrelated. Should the deviations of the measured points contain systematic local deviations of form of the surface, they are correlated, and the uncertainty may be calculated too big – but not too small [4-5, 7]. However, the number of points does not influence the other uncertainty components such as e.g. the geometrical error of the CMM, which depends on the accuracy of the CMM itself.

7. Discussion about the influence quantities

The uncertainty may be easily calculated and documented using a table calculation programme. The effects of the influence quantities may be thus tested and the measurement strategy be optimized, as it will be shown in the example of Table 1.
The probing of the surface of the object (D_W) is the biggest uncertainty component with $u_i(y)$=1.4 µm. Increasing the number of measured points to e.g. n=100, reduces it to $u_i(y)$=0.4 µm and will result in a combined standard uncertainty of $u_c(y)$=1.2 µm.
Now the probe system qualification (D_E) is the biggest uncertainty component with $u_i(y)$=0,8 µm. Increasing the number of points to e.g. n=25, it will be reduced to 0.4 µm. Both steps combined result in $u_c(y)$=1.0 µm. The biggest components now are those of the temperatures of the object and of the scales (0.6 rsp. 0.5 µm). Both may possibly be reduced further by bringing the object to the right temperature and measuring the temperatures more precisely [8].
The influence of the estimated geometrical errors of the CMM in comparison to the other components is very small and may be neglected. In the case of bigger diameters it actually increases proportionally, but the temperature influence increases, too. In the case of D=500 mm the uncertainty component of the temperature of the object will result in 2.9 µm and that of the scale in 2.1 µm and, finally, the combined standard uncertainty in $u_c(y)$=4.0 µm. The geometrical error, at 0.6 µm, remains negligible in comparison to the other components.
The extracted knowledge may be generalized (Table 2), distinguishing between the measurement of workpieces from mechanical engineering and the calibration of material standards.

8. Repeated Measurements

Repeated measurements are carried out whenever the relationship between the influence quantities and the measurand is not known, and the mathematical model cannot be stated. The best estimate $\bar{y}$ and the standard uncertainty of the measurand will then be calculated directly from the measured values y_i.
But in this way the different effects of the influence quantities cannot be determined. For example the temperature may be registered with each and every repeated measurement, but the errors of the workpiece are superposed by the random probing errors. Measuring small workpieces with a small number of points may often leave the influence of the temperature unrecognized.
A further problem is the varying measuring conditions. Because of organizational restrictions, repeated measurements are often carried out at short, successive intervals. But this does not, as it were, reflect the temperature variations as they occur throughout a day, week or seasons of the year. Therefore, the

evaluated uncertainty is only valid for the temperature range during measurements, which has to be documented with the uncertainty. For this reason the impact of the temperature may be better estimated according to the method B of the *GUM* [8, 11].

Table 2: Influence quantities to be taken into account in the mathematical model; the order represents their importance (value)

Dimensions in comparison to the CMM	Objects with relatively large local deviations of form (measurement of workpieces)	Objects with relatively small local deviations of form (calibration of material standards)
Small objects	• Number and position of the measured points on the surface	• Number and position of the measured points on the surface • Number and position of the measured points in probe system qualification • Temperatures of the object, the scales, the material standard and the stylus • Calibration of the material standard • Geometrical errors of the CMM • Time-dependent drift of the CMM
Large objects	• Temperatures of the object and the scales • Number and position of the measured points on the surface • Geometrical errors of the CMM	• Temperatures of the object, the scales, the material standard and the stylus • Number and position of the measured points on the surface • Number and position of the measured points in probe system qualification • Calibration of the material standard • Geometrical errors of the CMM • Time-dependent drift of the CMM

9. Local deviations of form

Repeated measurements may be carried out to evaluate the influence of local deviations of form on the surface, particularly if they are relatively large. But repeated measurements at the same points of the surface detect only the random probing errors (Fig. 4 above). Generally, the points to be measured are not defined in the technical design. That is why all points of the surface must have the same chance to be programmed and measured. The procedure of uncertainty evaluation has to take into account this random choice, and the dispersion of the results will become much bigger (Fig. 4 below).

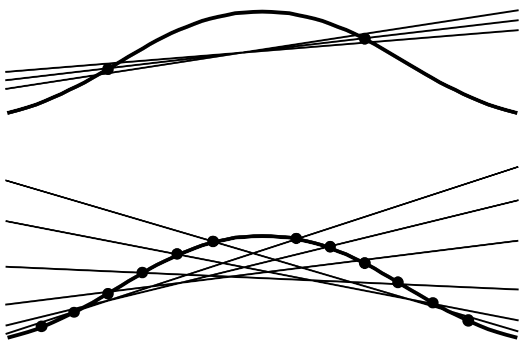


Fig. 4:
Measuring a straight line with two points at any one time; the effect of probing a surface with large local deviations of form at the same points (above) or at various points (below)

The standard deviation of the repeated measurements may then be used instead of the standard uncertainty $u(x_i)$ of the diameter D_W in Table 1. With a large number of measured points, it will come closer to the standard deviation of the probing system of the CMM. However, the influence of the temperatures has to be taken into account with each case. Changing them during the repeated measurements will increase the standard deviation.

10. Simulation

The uncertainty of measurement may also be evaluated by simulation, and initial experience with the Virtual CMM is documented [12]. Certainly, the mathematical models of the individual measuring tasks are not indicated. In this way neither the validity of the mathematical models may be verified nor may the effects of the individual influence quantities on the uncertainty be evaluated.
One further disadvantage of the implemented procedures is the one-sided concentration on the CMM itself. It takes some time to determine its errors when measuring material standards, e.g. ball plates, with a limited use, because the geometrical errors of the CMM may be easily estimated – provided that the maximum permissible error for size measurements is not exceeded. Also the influences of the temperatures may be estimated using the method B of the *GUM*.
Furthermore, the recently realized procedures of the Virtual CMM do not take into account at all the effect of the local deviations of form on the surface, neither how it may be estimated with the mathematical model nor how it might be evaluated by repeated measurements. That is why the Virtual CMM may be applied to the calibration of material standards, but not to measurements of workpieces with relevant deviations of form. In this case, this influence has to be quantified by other ways, and the Virtual CMM will be one component in a composite statement of uncertainty.

11. Substitution measurements

In substitution measurements, a working standard of similar shape and size to the object is to be measured in the same position and orientation on the CMM. The geometrical errors of the CMM are corrected by the measured deviations of the working standard. This method has already been known for a long time as the comparator method and is mainly applied to the calibration of working standards.
The mathematical model of a substitution measurement may generally be written as:

$$Y = X - X_S + X_C \qquad (8)$$

Using the standard deviations u_X, u_{XS} and u_{XC} of the influence quantities
- X Measurement of the object (workpiece),
- X_S Measurement of the working standard, and
- X_C Calibrated value of the material standard,

the combined standard uncertainty $u_c(y)$ of the measurand Y may easily be calculated to:

$$u_c(y) = \sqrt{u_x^2 + u_{xs}^2 + u_{xc}^2} \qquad (9)$$

The standard uncertainty u_{XC} of the calibrated value may be taken from the calibration certificate. The standard uncertainties u_X of the object's measurements and u_{XS} of the working standard may be evaluated e.g. by repeated measurements. In the case of the workpiece, the influence of the local deviations of form on the surface has to be taken into account by probing various points. Additionally, the influence of the temperature has to be calculated.
The measurement of the drill hole diameter documented in Table 1 may also be carried out as a substitution measurement using a calibrated gage ring instead of the spherical material standard. Such rings are usually calibrated with an expanded uncertainty of 0.7 to 1 µm. This corresponds to an uncertainty component of at least 0.35 µm. Measuring this ring with e.g. n=100 points and a standard deviation of s=1 µm results in an uncertainty component of 0.2 µm. Both components are bigger than the estimated component of the geometrical error of the CMM (0.1 µm). The substitution measurement yields no advantage. It would only increase the uncertainty of the drill hole diameter.
This is generally valid for all small workpieces, too. However, if the uncertainty components of the calibration of the working standard and its measurement are smaller than the estimated uncertainty component of the geometrical errors, the substitution measurement may be applied to reduce the uncertainty.
Recently, the substitution method is often used to calibrate gauges, because only small numbers of points are used in probe system qualification and in objects measurement. Increasing these numbers would be much more effective, in particular because spherical material standards may be much more precisely calibrated than gauge rings.

12. Conclusions

- The number and positions of measured points have a great effect on the uncertainty of coordinate measurements. This may be illustrated by the confidence intervals of the geometric features.
- The standard deviation of the measured points from the geometric features includes the impact of the local deviations of form and may be used to evaluate the uncertainty.
- The uncertainty component of the geometrical errors of the CMM may be estimated using the length-dependent part of the maximum permissible error MPE_E for size measurement.
- In repeated measurements of surfaces with relevant local deviations of form, in every measurement other points of the surface have to be probed.
- The effect of the temperature in repeated measurements often may not be determined completely and should rather be calculated.
- Simulating the geometrical errors of the CMM, the influence of the local deviations of form of the object is not taken into account and has to be quantified by other ways.
- Substitution measurements are only effective if the uncertainty can be really reduced.

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